Oxford Cambridge and RSA

GCE

Mathematics

Unit **4726**: Further Pure Mathematics 2

Advanced GCE

Mark Scheme for June 2015

This mark scheme is published as an aid to teachers and students, to indicate the requirements of the examination. It shows the basis on which marks were awarded by examiners. It does not indicate the details of the discussions which took place at an examiners' meeting before marking commenced.

All examiners are instructed that alternative correct answers and unexpected approaches in candidates' scripts must be given marks that fairly reflect the relevant knowledge and skills demonstrated.

Mark schemes should be read in conjunction with the published question papers and the report on the examination.

OCR will not enter into any discussion or correspondence in connection with this mark scheme.

Annotations and abbreviations

Annotation in scoris	Meaning
✓and ✗	
BOD	Benefit of doubt
FT	Follow through
ISW	Ignore subsequent working
M0, M1	Method mark awarded 0, 1
A0, A1	Accuracy mark awarded 0, 1
B0, B1	Independent mark awarded 0, 1
SC	Special case
^	Omission sign
MR	Misread
Highlighting	
Other abbreviations in mark scheme	**Meaning**
M1 dep	Method mark dependent on a previous mark
cao	Correct answer only
oe	Or equivalent
rot	Rounded or truncated
soi	Seen or implied
www	Without wrong working

Subject-specific Marking Instructions

a Annotations should be used whenever appropriate during your marking.

The A, M and B annotations must be used on your standardisation scripts for responses that are not awarded either 0 or full marks. It is vital that you annotate standardisation scripts fully to show how the marks have been awarded.

For subsequent marking you must make it clear how you have arrived at the mark you have awarded

b An element of professional judgement is required in the marking of any written paper. Remember that the mark scheme is designed to assist in marking incorrect solutions. Correct *solutions* leading to correct answers are awarded full marks but work must not be judged on the answer alone, and answers that are given in the question, especially, must be validly obtained; key steps in the working must always be looked at and anything unfamiliar must be investigated thoroughly.

Correct but unfamiliar or unexpected methods are often signalled by a correct result following an *apparently* incorrect method. Such work must be carefully assessed. When a candidate adopts a method which does not correspond to the mark scheme, award marks according to the spirit of the basic scheme; if you are in any doubt whatsoever (especially if several marks or candidates are involved) you should contact your Team Leader.

c The following types of marks are available.

M
A suitable method has been selected and *applied* in a manner which shows that the method is essentially understood. Method marks are not usually lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, eg by substituting the relevant quantities into the formula. In some cases the nature of the errors allowed for the award of an M mark may be specified.

A
Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated Method mark is earned (or implied). Therefore M0 A1 cannot ever be awarded.

B
Mark for a correct result or statement independent of Method marks.

Unless otherwise indicated, marks once gained cannot subsequently be lost, eg wrong working following a correct form of answer is ignored. Sometimes this is reinforced in the mark scheme by the abbreviation isw. However, this would not apply to a case where a candidate passes through the correct answer as part of a wrong argument.

d When a part of a question has two or more 'method' steps, the M marks are in principle independent unless the scheme specifically says otherwise; and similarly where there are several B marks allocated. (The notation 'dep *' is used to indicate that a particular mark is dependent on an earlier, asterisked, mark in the scheme.) Of course, in practice it may happen that when a candidate has once gone wrong in a part of a question, the work from there on is worthless so that no more marks can sensibly be given. On the other hand, when two or more steps are successfully run together by the candidate, the earlier marks are implied and full credit must be given.

e The abbreviation ft implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A and B marks are given for correct work only — differences in notation are of course permitted. A (accuracy) marks are not given for answers obtained from incorrect working. When A or B marks are awarded for work at an intermediate stage of a solution, there may be various alternatives that are equally acceptable. In such cases, exactly what is acceptable will be detailed in the mark scheme rationale. If this is not the case please consult your Team Leader.

Sometimes the answer to one part of a question is used in a later part of the same question. In this case, A marks will often be 'follow through'. In such cases you must ensure that you refer back to the answer of the previous part question even if this is not shown within the image zone. You may find it easier to mark follow through questions candidate-by-candidate rather than question-by-question.

f Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise. Candidates are expected to give numerical answers to an appropriate degree of accuracy, with 3 significant figures often being the norm. Small variations in the degree of accuracy to which an answer is given (e.g. 2 or 4 significant figures where 3 is expected) should not normally be penalised, while answers which are grossly over- or under-specified should normally result in the loss of a mark. The situation regarding any particular cases where the accuracy of the answer may be a marking issue should be detailed in the mark scheme rationale. If in doubt, contact your Team Leader.

g Rules for replaced work

If a candidate attempts a question more than once, and indicates which attempt he/she wishes to be marked, then examiners should do as the candidate requests.

If there are two or more attempts at a question which have not been crossed out, examiners should mark what appears to be the last (complete) attempt and ignore the others.

NB Follow these maths-specific instructions rather than those in the assessor handbook.

h For a *genuine* misreading (of numbers or symbols) which is such that the object and the difficulty of the question remain unaltered, mark according to the scheme but following through from the candidate's data. A penalty is then applied; 1 mark is generally appropriate, though this may differ for some units. This is achieved by withholding one A mark in the question.

Note that a miscopy of the candidate's own working is not a misread but an accuracy error.

Question			Answer	Marks	Guidance	
1			$\tanh^{-1} x = y \Rightarrow x = \tanh y = \dfrac{e^{2y}-1}{e^{2y}+1}$	**M1**	**Oe**	A muddle of x and y unless recovered is M0.
			$\left(e^{2y}+1\right)x = e^{2y}-1$			
			$e^{2y}\left(1-x\right) = \left(1+x\right)$			
			$\Rightarrow e^{2y} = \dfrac{1+x}{1-x}$	**A1**	Correct expression for e^{2y} **oe**	
			$2y = \ln\left(\dfrac{1+x}{1-x}\right)$			
			$\left(y = \tanh^{-1} x\right) = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$	**A1**	**ag**	
				3		

Question			Answer	Marks	Guidance	
2			$\ln(1+y)=y-\frac{y^2}{2}+\frac{y^3}{3}-...$	**B1**	Soi. Allow an expansion in x	
			$\sin x = x-\frac{x^3}{6}+...$	**B1**	Soi	
			$\ln(1+\sin x)=\left(x-\frac{x^3}{6}\right)-\frac{1}{2}\left(x-\frac{x^3}{6}\right)^2+\frac{1}{3}\left(x-\frac{x^3}{6}\right)^3-...$ $=x-\frac{1}{2}x^2+x^3\left(\frac{1}{3}-\frac{1}{6}\right)$	**M1**	For combining series, even if wrong. Must include at least the cubic bracket.	
			$=x-\frac{1}{2}x^2+\frac{1}{6}x^3$	**A1**	Ignore further terms www accept 3! for 6	
				4		
			Alternative using Maclaurin general formula $f(x)=\ln(1+\sin x)$ $\quad f(0)=0$			
			$f'(x)=\frac{\cos x}{(1+\sin x)}$ $\quad f'(0)=1$	**B1**	For $f'(x)$	
			$f''(x)=\frac{-1}{(1+\sin x)}$ $\quad f''(0)=-1$	**B1**	For (not necessarily simplified) $f''(x)$ and $f''(0)$ www	
			$f'''(x)=\frac{\cos x}{(1+\sin x)^2}$ $\quad f'''(0)=1$			
			Maclaurin: $f(x)=f(0)+f'(0)x+\frac{f''(0)x^2}{2}+\frac{f'''(0)x^3}{6}$	**M1**	For correct formula up to 4th term and substituting *their* values	
			$\Rightarrow f(x)=x-\frac{1}{2}x^2+\frac{1}{6}x^3$	**A1**	Accept 3! for 6	

Question			Answer	Marks	Guidance	
3			$\int_{\frac{1}{2}}^{1} \frac{1}{\sqrt{2x-x^2}}\,\mathrm{d}x = \int_{\frac{1}{2}}^{1} \frac{1}{\sqrt{1+2x-x^2-1}}\,\mathrm{d}x$	M1	Completing the square on given function	Or
			$= \int_{\frac{1}{2}}^{1} \frac{1}{\sqrt{1-(1-x)^2}}\,\mathrm{d}x$	A1		$= \int_{\frac{1}{2}}^{1} \frac{1}{\sqrt{1-(x-1)^2}}\,\mathrm{d}x$
			$= \left[-\sin^{-1}(1-x)\right]_{\frac{1}{2}}^{1}$	M1	By substitution or using standard form where completed square is of form $1-(1\pm x)^2$	$= \left[\sin^{-1}(x-1)\right]_{\frac{1}{2}}^{1} = \left(0 - -\frac{\pi}{6}\right) = \frac{\pi}{6}$
				A1	Correct result of integration. Ignore limits	
			$= -\left(0 - \frac{\pi}{6}\right) = \frac{\pi}{6}$	A1		
				5		

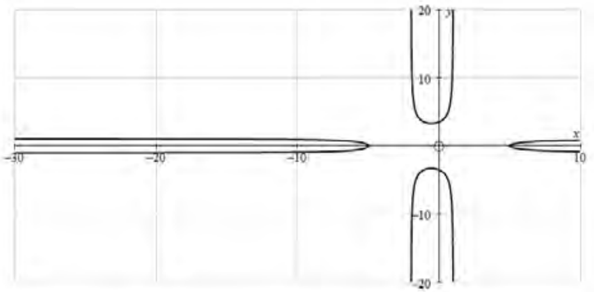
Question			Answer	Marks	Guidance	
4	(i)		$I_n = \int_0^1 x^n e^{-x}\,dx \quad u = x^n \quad dv = e^{-x}\,dx$ $du = nx^{n-1}dx \quad v = -e^{-x}$ $I_n = \left[-e^{-x}x^n\right]_0^1 + n\int_0^1 x^{n-1}e^{-x}\,dx$ $= \left(-e^{-1} - 0\right) + nI_{n-1}$ $I_n = nI_{n-1} - e^{-1}$	M1 A1 A1	By parts Both terms before limits are applied soi Or $k = \frac{-1}{e}$	
				3		
	(ii)		$I_0 = \int_0^1 e^{-x}dx = \left[-e^{-x}\right]_0^1 = 1 - e^{-1}$ $I_3 = 3I_2 - e^{-1}$ $= 3\left(2I_1 - e^{-1}\right) - e^{-1} = 6I_1 - 4e^{-1}$ $= 6(I_0 - e^{-1}) - 4e^{-1} = 6I_0 - 10e^{-1}$ $I_3 = 6 - 16e^{-1}$	B1 M1 A1	Complete method even if k is wrong. **SC3** by parts 2 or 3 times	Or finding I_1. Or could be done the other way round.
				3		
	(iii)		$I_{11} = 11I_{10} - e^{-1}$ $= 11\left(10I_9 - e^{-1}\right) - e^{-1} = 110I_9 - 12e^{-1}$ $= 110(9I_8 - e^{-1}) - 12e^{-1} = 990I_8 - 122e^{-1}$ $990I_8 - I_{11} = 122e^{-1}$	M1 A1 A1	Complete method (Could be done the other way round.) For I_{11} in terms of I_{10} or I_9 in terms of I_8 soi	Alternative: Starting from I_4 and working up to I_{11} **M1** I_8 or I_{11} correct **A1** $I_8 = 8! - \frac{109601}{e}, \quad I_{11} = 11! - \frac{108505112}{e}$
				3		

Question			Answer	Marks	Guidance	
5	(i)		$y=\sin^{-1}(2x)\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{\sqrt{1-(2x)^2}}.\frac{\mathrm{d}(2x)}{\mathrm{d}x}$ $=\frac{2}{\sqrt{1-4x^2}}$	B1	**Oe**	
				1		
	(ii)		$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2\times\left(-\frac{1}{2}\right)\left(1-4x^2\right)^{-\frac{3}{2}}(-8x)=\frac{8x}{\left(1-4x^2\right)^{\frac{3}{2}}}$ $=\frac{8x}{\left(1-4x^2\right)\sqrt{1-4x^2}}=\frac{4x}{\left(1-4x^2\right)}\frac{\mathrm{d}y}{\mathrm{d}x}$ $\left(1-4x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=4x\frac{\mathrm{d}y}{\mathrm{d}x}$	B1 M1 A1	For correct 2nd derivative Using *their* ans to connect 1st and 2nd derivatives Ft to achieve ag	**SC 2** if result obtained correctly from $y'=\frac{k}{\sqrt{\left(1-4x^2\right)}}$
				3		
	(iii)		$\left(1-4x^2\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}-8x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=4\frac{\mathrm{d}y}{\mathrm{d}x}+4x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ $\left(1-4x^2\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}-12x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-4\frac{\mathrm{d}y}{\mathrm{d}x}=0$	M1 A1	Using result of (ii) and product rule correctly	**M1** Starting with *their* 2nd derivative using appropriate method correctly **A1** ans www
				2		
	(iv)		Find $y_0, y'_0,\ y''_0,\ y'''_0=\{0,2,0,8\}$ $y=0+2x+0+\frac{8x^3}{6}$ $\Rightarrow y=2x+\frac{4x^3}{3}$	B1 M1 A1	soi Correctly substituting *their* 4 values into correct Maclaurin www Ignore higher order terms	
				3		

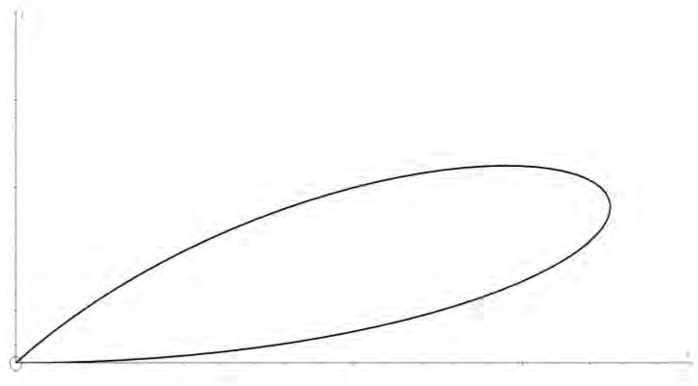
Question			Answer	Marks	Guidance	
6	(i)		$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}=x_n-\frac{3x_n^3+5x_n^2-x_n-1}{9x_n^2+10x_n-1}$	B1	Correct derivative seen	
			$=\frac{x_n(9x_n^2+10x_n-1)-(3x_n^3+5x_n^2-x_n-1)}{9x_n^2+10x_n-1}$	M1	Combining terms seen as 1 fraction or 2 fractions with common denominator	
			$=\frac{9x_n^3+10x_n^2-x_n-3x_n^3-5x_n^2+x_n+1}{9x_n^2+10x_n-1}$			
			$=\frac{6x_n^3+5x_n^2+1}{9x_n^2+10x_n-1}$	A1	Line above seen ag Must contain suffices.	
				3		
	(ii)		A suitable value is shown within range [0.1, 0.25]	B1	The point does not have to be labelled x_1	Accept a tangent which shows this.
				1		
	(iii)		$\Rightarrow x_2=0\Rightarrow x_3=-1$, and statement that values alternate.	B1	Values seen either in words or on graph marked as these values	
			Clear diagram with tangents from – 1 to 0 and back to –1	B1		
				2		
	(iv)		$d_4=kd_3^2,\quad d_3=kd_2^2$	M1	d_4 and d_3 and trying to combine them to eliminate k	
			$\Rightarrow\frac{d_4}{d_3}=\frac{kd_3^2}{kd_2^2}=\frac{d_3^2}{d_2^2}\Rightarrow d_4=\frac{d_3^3}{d_2^2}$	A1	Ag	
			(see table below)	B1	Sight of –0.0300	Condone 3 dp
				B1	Sight of –0.0358	3sf or better

x_r	x_{r+1}	d_r		$\frac{d_3^3}{d_2^2}$
1	0.666667	-0.33333	d_2	
0.666667	0.517241	-0.14943	d_3	
0.517241	0.481438	-0.0358	d_4	-0.030
0.481438	0.479363	-0.00207	d_5	
0.479363	0.479356	-6.8E-06		
0.479356	0.479356			

Question			Answer	Marks	Guidance	
				4		
	(v)		Continuing the above to give root 0.47936	**M1** **A1**	Or any other starting point that converges to the positive root Cao	
				2		

Question			Answer	Marks	Guidance	
7	(i)		$\frac{x^2-25}{(x-1)(x+2)} = A + \frac{B}{(x-1)} + \frac{C}{(x+2)}$ $x^2-25 = A(x-1)(x+2)+B(x+2)+C(x-1)$ 3 processes of equating coefficients or substituting: e.g. $x=1 \Rightarrow -24=3B \Rightarrow B=-8$ $x=-2 \Rightarrow -21=-3C \Rightarrow C=7$ coeff of x^2 : $A=1$ $\frac{x^2-25}{(x-1)(x+2)} = 1 - \frac{8}{(x-1)} + \frac{7}{(x+2)}$	**M1** **B1** **A1** **A1**	Splitting in correct way to give partial fractions (may be seen anywhere) For A For B For C	
				4		
	(ii)		$x=1, x=-2$ $y=1$	**B1** **B1**		
				2		
	(iii)		$y=1 \Rightarrow (x-1)(x+2)=x^2-25$ $x^2+x-2=x^2-25 \Rightarrow x=-23$	**M1** **A1**		
				2		
	(iv)			**B1** **B1**	4 bits as shown, roughly symmetric about axes, approaching asymptotes Lh side crosses asymptotes and upper section approaches from above and lower section approaches from below	Ignore any graph of $y = \mathrm{f}(x)$
				2		

Question			Answer	Marks	Guidance	
8	(i)		$y = 2\sinh x + 3\cosh x \Rightarrow \frac{dy}{dx} = 2\cosh x + 3\sinh x$	M1	Diffn and setting = 0	$y = \frac{2}{2}\left(e^x - e^{-x}\right) + \frac{3}{2}\left(e^x + e^{-x}\right) = \frac{1}{2}\left(5e^x + e^{-x}\right)$
			$= 0$ when $2\cosh x = -3\sinh x \Rightarrow \tanh x = -\frac{2}{3}$	A1	Correct value for sinh*x*, cosh*x* or tanh*x*	$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(5e^x - e^{-x}\right)$
			$x = \tanh^{-1}\left(-\frac{2}{3}\right) = \frac{1}{2}\ln\left(\frac{1-\frac{2}{3}}{1+\frac{2}{3}}\right) = \frac{1}{2}\ln\left(\frac{1}{5}\right) = -\frac{1}{2}\ln 5$	A1	some numerical justification must be seen ag	$= 0$ when $5e^x = e^{-x} \Rightarrow e^{2x} = \frac{1}{5} \Rightarrow x = -\frac{1}{2}\ln 5$ Correct exponential form, diffn, set = 0 A1 correct e^{2x} A1 answer
			$\sinh x = \frac{-2}{\sqrt{5}}, \cosh x = \frac{3}{\sqrt{5}} \Rightarrow y = \frac{-4}{\sqrt{5}} + \frac{9}{\sqrt{5}} = \sqrt{5}$	B1	Exact answer only	**SC** Substitute given value of *x* into derivative to get 0 is 1/3
				4		
	(ii)		$2\sinh x + 3\cosh x = 5 \Rightarrow 2\frac{e^x - e^{-x}}{2} + 3\frac{e^x + e^{-x}}{2} = 5$	M1	Find exponential form	Alt: $\Rightarrow \sqrt{5}\cosh(x+\alpha) = 5$ where $\alpha = \frac{1}{2}\ln 5$
			$5e^x + e^{-x} = 10$ $5e^{2x} - 10e^x + 1 = 0$	A1	Correct quadratic	$\Rightarrow x = \ln\left(\frac{2+\sqrt{5}}{\sqrt{5}}\right)$ and $-\ln\left(2\sqrt{5}+5\right)$
			$e^x = \frac{10 \pm \sqrt{100-20}}{10} = \frac{10 \pm \sqrt{80}}{10}$	M1	Solve *their* 3 term quadratic	
			$x = \ln\left(1 + \frac{2\sqrt{5}}{5}\right)$ and $\ln\left(1 - \frac{2\sqrt{5}}{5}\right)$	A1 A1	**oe** Single ln only	Penalise only once
				5		

Question			Answer	Marks	Guidance	
9	(i)			B1 B1	Enclosed loop in first quadrant with origin as pole Looking symmetric with line of symmetry around $\theta = \frac{\pi}{6}$ Take one off full marks for more loops	N.B. This means that $\theta = \frac{\pi}{2}$ is not a tangent at the pole.
				2		
	(ii)		Area $= \frac{1}{2}\int_0^{\pi/3} r^2 \mathrm{d}\theta = 2\int_0^{\pi/3} \sin^2 3\theta \, \mathrm{d}\theta$ $= \int_0^{\pi/3} (1 - \cos 6\theta)\mathrm{d}\theta = \left[\theta - \frac{1}{6}\sin 6\theta\right]_0^{\pi/3}$ $= \frac{\pi}{3}$	M1 M1 A1 A1	Correct formula plus limits For obtaining fn in form to integrate using double angle formulae Integral Ft lack of $\frac{1}{2}$ Answer www	Must include $\frac{1}{2}$
				4		
			Alternative: Starting from given equation: Eliminating x and y M1 Get r M1			

Question			Answer	Marks	Guidance	
	(iii)		$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$	**M1**	Obtaining $\sin 3\theta$ as a function of θ	
			$y = r\sin\theta \Rightarrow \sin\theta = \frac{y}{r}$ and $r^2 = x^2 + y^2$	**A1**	A correct expression	
			$r = 2\sin 3\theta = 6\sin\theta - 8\sin^3\theta = \frac{6y}{r} - \frac{8y^3}{r^3}$	**M1**	Eliminate θ	
			$\Rightarrow r^4 = 6yr^2 - 8y^3$	**M1**	Eliminate r	
			$\Rightarrow \left(x^2 + y^2\right)^2 = 6\left(x^2 + y^2\right)y - 8y^3 = 6x^2y - 2y^3$	**A1**	ag	
				5		
			Alternative: Starting from given equation: Eliminating x and y **M1** Get r **M1** $r = 6\sin\theta - 8\sin^3\theta$ **A1** Obtain triple angle formula **M1** Ans **A1**			

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